

On the Energy of Charged Black Holes in Generalized Dilaton-Axion Gravity

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Abstract In this paper we calculate the energy distribution of some charged black holes in generalized dilaton-axion gravity. The solutions correspond to charged black holes arising in a Kalb-Ramond-dilaton background and some existing non-rotating black hole solutions are recovered in special cases. We focus our study to asymptotically flat and asymptotically non-flat types of solutions and resort for this purpose to the Møller prescription. Various aspects of energy are also analyzed.

Keywords Energy · Charged black-hole · Dilaton-axion gravity

1 Introduction

In the recent years, a wide interest have been focused on numerous efficient and precise tools, such as superenergy-tensors [1–16], energy-momentum complexes, quasi-local expressions [17–20] and the tele-parallel theory of gravitation [21–35] for the study of energy-momentum localization.

In General Relativity, the problem of localization of energy using energy-momentum complexes was discussed first by Einstein who constructed his pseudotensor [36, 37], and other prescriptions were elaborated later by Landau-Lifshitz [38], Papapetrou [39], Bergmann-Thompson [40], Weinberg [41], Qadir-Sharif [42] and Møller [43]. Among these prescriptions, the Møller definition is the only one which can be applied to any coordinate

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system, since the other energy-momentum complexes generate meaningful results only in the case of the quasi-Cartesian coordinates. Light has been shed upon the topic of energy-momentum localization in the last two decades and the pseudotensorial definitions have also been employed for computing the energy in the case of some 2 + 1 and 2 dimensional space-times, emphasizing the fact that different pseudotensorial definitions can yield the same expression for the energy distribution of a given space-time [44–67]. We may thus notice that, in many cases, the energy-momentum complexes produce the same results as their teleparallel versions [68–70]. Virbhadra came up with an important result and proved that using different energy-complexes (ELLPW) it is possible to obtain the same result for a general non-static spherically symmetric metric of the Kerr-Schild class [71–75]. In addition, these definitions (ELLPW) are compliant with the quasi-local mass definition given by Penrose [76] and verified by Tod [77] in the case of a general non-static spherically symmetric metric of the Kerr-Schild class. Nevertheless, these definitions disagree for the most general non-static spherically symmetric metric (Virbhadra [71]). We should also mention the significant results obtained by several authors with the Møller prescription [78–115]. Moreover, this definition is considered from the viewpoint of Lessner [116] as an accurate and powerful tool for energy localization in General Relativity. Supporting the Lessner opinion and the meaningful results obtained by several researchers, Chang, Nester and Chen [117] stressed the fact that the energy-momentum complexes are quasilocal expressions for energy-momentum. They reached the conclusion that these pseudotensorial definitions and the quasilocal expressions are connected in a direct manner, and that every energy-momentum complex is associated with a legitimate Hamiltonian boundary term. Furthermore, each expression for energy has a geometrical and physical significance due to the connection with the boundary conditions. All these assumptions emphasize the significance of the energy-momentum complexes and point out their usefulness for the energy-momentum localization.

In this paper, using the Møller prescription we calculate the energy distribution of charged black holes in generalized dilaton-axion gravity inspired by low energy string theory.

The remainder of our paper is organized as follows: in Sect. 2 we present an overview of the space-time under consideration which describes new black hole solutions for the Einstein-Maxwell scalar field system inspired by low energy string theory [118]. These solutions have an electric and a magnetic charge and some non-rotating black hole solutions are obtained in special limit cases. The Møller energy-momentum complex is described in Sect. 3. This section is also devoted to the evaluation of the momenta and energy distributions, and to the analysis of various aspects of energy. Finally, our concluding remarks are drawn in Discussion. For our calculations we consider the signature (1, -1, -1, -1), geometrized units ($c = 1$; $G = 1$) and assume that Greek (Latin) indices take value from 0 to 3 (1 to 3).

2 Charged Black Holes Generated in Einstein-Maxwell-Dilaton-Axion Theory

Recently, S. Sur, S. Das and S. SenGupta [118] have discovered new black hole solutions for Einstein-Maxwell scalar field system inspired by low energy string theory. They considered the action in which two scalar fields are minimally coupled to Einstein-Hilbert-Maxwell field in the Einstein frame in four dimension as

$$I = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\omega(\phi)}{2} \partial_\mu \xi \partial^\mu \xi - \alpha(\phi, \xi) F^2 - \beta(\phi, \xi) F_{\mu\nu} {}^* F^{\mu\nu} \right], \quad (1)$$

where $\kappa = 8\pi G$, R represents the curvature scalar, $F_{\mu\nu}$ is the Maxwell field tensor, F is the contracted Maxwell scalar i.e. $F^\mu_\mu = F$ while ϕ and ξ are two massless scalar or pseudo scalar fields which are coupled to Maxwell field with the functional relationship α and β . Here, ξ acquires a non minimal kinetic term ω . In the context of low energy string theory, the fields ϕ and ξ can be identified as massless scalar dilaton and pseudoscalar axion, respectively. Two other important quantities are the effective scalar field $\psi(r)$ that is defined in terms of ϕ and ξ as $\psi'^2 = \phi'^2 + \omega \xi'^2$, and the effective coupling $\gamma(r)$. Sur et al. [118] have found a most general class of static spherically symmetric black hole solutions classified as asymptotically flat and asymptotically non-flat types (Sect. 4 in [118]).

Considering a generalized form of the above action in (1), with the corresponding connections $\omega(\phi) = e^{2a\phi}$, $\alpha(\phi) = e^{-a\phi}$ and $\beta(\xi) = b\xi$ where a is a real constant which is also non-negative, Sur et al. [118] have analyzed their solutions in the context of the low energy effective string theory (Sect. 5 in [118]).

We present the asymptotically flat and the asymptotically non-flat black holes solutions obtained by Sur et al. [118] and which are in general electrically and magnetically charged.

For asymptotically flat black holes the metric is given by

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - h(r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where

$$f(r) = \frac{(r - r_-)(r - r_+)}{(r - r_0)^{(2-2n)}(r + r_0)^{2n}}, \tag{3}$$

and

$$h(r) = \frac{(r + r_0)^{2n}}{(r - r_0)^{(2n-2)}}. \tag{4}$$

with $0 < n < 1$ and r_0 is a constant real parameter. Also other various parameters are given by

$$\begin{aligned} r_{\pm} &= m_0 \pm \sqrt{m_0^2 + r_0^2 - \frac{1}{8} \left(\frac{K_1}{n} + \frac{K_2}{1-n} \right)}, \\ r_0 &= \frac{1}{16m_0} \left(\frac{K_1}{n} - \frac{K_2}{1-n} \right), \\ m_0 &= m - (2n - 1)r_0, \\ K_1 &= 4n[4r_0^2 + 2kr_0(r_+ + r_-) + k^2r_+r_-], \\ K_2 &= 4(1 - n)r_+r_-, \quad 0 < n < 1, \\ m &= \frac{1}{16r_0} \left(\frac{K_1}{n} - \frac{K_2}{1-n} \right) + (2n - 1)r_0. \end{aligned} \tag{5}$$

Here m is the mass of the black hole. The effective scalar is defined as

$$\psi(r) = \psi_0 + 2\sqrt{n(n-1)} \ln\left(\frac{r-r_0}{r+r_0}\right) \tag{6}$$

and the effective coupling is given by

$$\gamma(r) = K_1 \left(\frac{r-r_0}{r+r_0} \right)^{2(1-n)} + K_2 \left(\frac{r-r_0}{r+r_0} \right)^{-2n}. \tag{7}$$

After performing some calculations the total (bare) electric and magnetic charges Q_e and Q_m are found to be connected to the scalar field shielded electric and magnetic charges q_e and q_m through the relations

$$Q_e = (q_e - q_m b \xi_0) e^{\alpha \phi_0}, \quad Q_m = q_m \tag{8}$$

and the electromagnetic field strengths F_{1r} and $F_{\theta\varphi}$ are given by

$$F_{1r} = \frac{[Q_e e^{-\alpha \phi_0} - Q_m b (\xi - \xi_0)] e^{\alpha \phi}}{(r - r_0)^{2(1-n)} (r + r_0)^{2n}} dt \wedge dr, \quad F_{\theta\varphi} = Q_m \sin \theta d\theta \wedge d\varphi. \tag{9}$$

The asymptotically non-flat black holes are obtained for

$$f(r) = \frac{(r - r_-)(r - r_+)}{r^2 (2r_0/r)^{2n}}, \tag{10}$$

$$h(r) = r^2 \left(\frac{2r_0}{r} \right)^{2n}, \tag{11}$$

$$r_{\pm} = \left(\frac{1}{1-n} \right) \left[m \pm \sqrt{m^2 - (1-n) \frac{K^2}{4}} \right] \tag{12}$$

and for the $\psi(r)$ and $\gamma(r)$ given by

$$\psi(r) = \psi_0 - 2\sqrt{n(n-1)} \ln \left(\frac{2r_0}{r} \right), \tag{13}$$

$$\gamma(r) = (4nr^2 + K_2) \left(\frac{2r_0}{r} \right)^{2n}. \tag{14}$$

The presence of the parameters a and b in the generalized action for Einstein-Maxwell theory in four dimensions, coupled to the massless scalar dilaton ϕ and the massless pseudoscalar axion ξ in Einstein frame has two motivations. The role of the parameter a is to be a regulator for the strength of the coupling between the dilaton and the Maxwell field. The parameter b is connected with the Kalb-Ramond tensor $H_{\mu\nu\lambda}$ which appears in the four dimensional heterotic string action [118] (see (5.3) therein). Another explanation for the introduction of the parameters a and b is that for some specific values the generalized action (see (5.1) in [118]) yields the field equations which correspond to a four dimensional effective compactified version of a higher dimensional (bulk) Einstein-Maxwell-Kalb-Ramond theory in a Randall-Sundrum scenario that is connected to the Planck-electroweak hierarchy problem. Some particular values of the parameters a and b lead to special cases, for $a = 1$ the field theoretic limit in the case of the ten dimensional or the effective four dimensional superstring model, in the bosonic sector is reached. For $a = \sqrt{1 + 2/n}$ the four dimensional Kaluza-Klein toroidal reduction of a $4 + n$ dimensional theory is recovered. The case of usual Einstein-Maxwell theory that is coupled minimally with a massless Klein-Gordon scalar field ϕ is obtained for $a = 0$ ignoring the presence of the other scalar ξ (or the KR tensor $H_{\mu\nu\lambda}$).

The effective field equations obtained for the general formalism take a new form. Solving these equations for the asymptotically flat and asymptotically non-flat black holes and imposing some specific values for the parameters a and b the expressions for $\phi(r)$ and $\xi(r)$ are determined.

In this paper we evaluate the energy and momentum distributions in the Møller prescription for asymptotically flat (AF) and asymptotically non-flat (ANF) solutions in the context of low energy string theory. Taking into account two special values as $|b| = |a|$ and $|b| \neq |a|$ with some particular cases for the parameters a and b we also analyze various aspects of the energy distribution.

3 Energy and Momentum in the Møller Prescription

We perform the calculations in the Møller prescription in the Einstein frame applying this definition to the metrics given by (2), (3), (4), (10) and (11) because we don't need to carry out the calculations in quasi-Cartesian coordinates. Next, we briefly revise the expressions for the Møller energy-momentum complex Θ_ν^μ , the Møller superpotential $M_\nu^{\mu\lambda}$, the energy density Θ_0^0 and the momentum density Θ_i^0 components, and also the expressions for the energy and momentum P_μ .

The Møller energy-momentum complex [43] is given by the definition

$$\Theta_\nu^\mu = \frac{1}{8\pi} M_{\nu, \lambda}^{\mu\lambda}, \tag{15}$$

where $M_\nu^{\mu\lambda}$ represents Møller's superpotential

$$M_\nu^{\mu\lambda} = \sqrt{-g} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma}. \tag{16}$$

The Møller superpotential is antisymmetric

$$M_\nu^{\mu\lambda} = -M_\nu^{\lambda\mu}. \tag{17}$$

The Møller energy-momentum complex holds the local conservation law

$$\frac{\partial \Theta_\nu^\mu}{\partial x^\mu} = 0, \tag{18}$$

where Θ_0^0 and Θ_i^0 represent the energy density and the momentum density components, respectively.

The energy and momentum are given by

$$P_\mu = \int \int \int \Theta_\mu^0 dx^1 dx^2 dx^3. \tag{19}$$

For the metric given by (2) the components of the Møller superpotential have the following expressions

$$M_0^{01} = h(r) \frac{\partial f(r)}{\partial r} \sin \theta, \tag{20}$$

$$M_2^{21} = f(r) \frac{\partial h(r)}{\partial r} \sin \theta, \tag{21}$$

$$M_3^{31} = f(r) \frac{\partial h(r)}{\partial r} \sin \theta, \tag{22}$$

$$M_3^{32} = 2 \cos \theta. \tag{23}$$

The equations (20)–(23) present a dependence on the metric functions $f(r)$ and $h(r)$, on their first derivative with respect to r coordinate $\frac{\partial f(r)}{\partial r}$ and $\frac{\partial h(r)}{\partial r}$, and on θ coordinate through $\sin\theta$ and $\cos\theta$. The expression for energy in the case of a nonstatic spherically symmetric metrics was calculated in [80]. For the metrics described by (2)–(4) and (2), (10), (11) all the momenta vanish. Using (19) and (20) we can calculate the expressions for energy.

We return to the asymptotically flat and asymptotically non-flat black hole solutions and perform our study considering the special values $|b| = |a|$ and $|b| \neq |a|$ and some particular cases for the parameters a and b . In the asymptotic limit the connections between $\phi, \xi, \phi', \xi', K_1, K_2, q_e, q_m, Q_e, Q_m, a, b, r, r_+, r_-, r_0$ and $Q^2 = Q_e^2 + Q_m^2$ are given in [118] (see (5.14) and (5.15) therein).

(1) Firstly, we present the results for the asymptotically flat black hole solutions.

Case I. $|b| = |a|$

The equations (5.15) in [118] are satisfied uniquely for the values $n = 1/(1 + a^2)$ and $K_2 = 0$, leading to the following expressions for r_0, m_0, r_+ and r_-

$$r_0 = \frac{(1 + a^2)Q^2 e^{-\alpha\phi_0}}{4m_0}, \tag{24}$$

$$m_0 = m - \frac{(1 - a^2)}{(1 + a^2)}r_0 \quad (Q^2 = Q_e^2 + Q_m^2), \tag{25}$$

$$r_+ = 2m_0 - r_0, \tag{26}$$

$$r_- = r_0. \tag{27}$$

Performing a coordinate shift $r + r_0 \rightarrow r$ the metric described by (2), (3) and (4) can be written in a new form

$$ds^2 = \left(1 - \frac{2m_0}{r}\right)\left(1 - \frac{2r_0}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 - \left(1 - \frac{2m_0}{r}\right)^{-1} \left(1 - \frac{2r_0}{r}\right)^{\frac{a^2-1}{a^2+1}} dr^2 - r^2 \left(1 - \frac{2r_0}{r}\right)^{\frac{2a^2}{1+a^2}} (d\theta^2 + \sin^2\theta d\varphi^2). \tag{28}$$

The dilaton field $\phi(r)$, the axion field $\xi(r)$ and the electromagnetic field strengths F_{tr} and $F_{\theta\varphi}$ are expressed by (5.20) and (5.21) of [118] with r_0 and m_0 given by (24) and (25).

Using (15) and (19) we obtain that the expression for energy in the Møller prescription is given by

$$E(r) = \frac{m_0 a^2 r + m_0 r - 4m_0 r_0 - r_0 a^2 r + r_0 r}{r(a^2 + 1)}. \tag{29}$$

From (29) we notice that the energy distribution depends on the parameters m_0, a, r_0 and r .

There are 3 particular limiting cases that we present in the following.

a. For $a = b = 1$ we lead to the bosonic sector of the ten dimensional heterotic superstring toroidally compactified to four spacetime dimensions. The metric given by (28) and the dilaton and axion fields have a new form [118] (see (5.23) and (5.24) therein) and the energy is

$$E(r) = m \left(1 - \frac{2r_0}{r}\right) = m - \frac{Q^2 e^{-\phi_0}}{r}, \tag{30}$$

where $r_0 = Q^2 e^{-\phi_0} / (2m)$. If $Q_e = 0, Q_m = Q$ or $Q_m = 0, Q_e = Q$ we recover the solutions given by Garfinkle, Horowitz and Strominger (GHS) [119, 120] and Gibbons [121] and explained by Gibbons and Maeda (GM) [122] (the solutions are elaborated in [119, 120] and [121] assuming a zero value or at least a trivial value for the KR axion field). These solutions describe a magnetically or electrically charged dilaton black hole. The non-trivial dilaton-axion configuration can be obtained using a magnetically (or, electrically) charged dilaton black hole configuration with the help of the $SL(2, R)$ invariance, even when the value of the parameter $a \neq 1$.

b. In the case $a = b \ll 1$, after some calculations [118] (solving (5.20) therein using $r_0 = Q^2 / (4m_0) + O(a^2)$), is demonstrated that the black hole solutions are characterized by the parameters

$$\phi(r) = \phi_0 + \frac{4ar_0}{r} \left(\frac{Q_m^2 - Q_e^2}{Q^2} \right) + O(a^3), \quad \xi(r) = \xi_0 + \frac{4ar_0}{r} \left(\frac{Q_m Q_e}{Q^2} \right) + O(a^3), \tag{31}$$

$$r_0 = \frac{1}{2}(m - \sqrt{m^2 - Q^2}) + O(a^2), \quad m_0 = \frac{1}{2}(m + \sqrt{m^2 - Q^2}) + O(a^2). \tag{32}$$

For $a \rightarrow 0$ this is the case of the standard dyonic Reissner-Nordstöm black hole solution. Using (29) and (32) the energy becomes

$$E(r) = m - \frac{Q^2}{r}. \tag{33}$$

c. For $a = b \gg 1$ the parameters r_0 and m_0 have the expressions [118]

$$r_0 \approx \frac{a^2 Q^2 e^{-\alpha\phi_0}}{4m_0}, \quad m_0 \approx m + r_0. \tag{34}$$

Considering that in the limit $a \rightarrow \infty$ the constants r_0 and m_0 could not be larger than m and after some calculations the dilaton and axion fields, respectively are given by (5.29) in [118] and the metric is

$$ds^2 = \left(1 - \frac{2m}{r - 2r_0} \right) dt^2 - \left(1 - \frac{2m}{r - 2r_0} \right)^{-1} dr^2 - (r - 2r_0)^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{35}$$

With a coordinate changing in $r - 2r_0 = r$ the standard Schwarzschild black hole is obtained together with non-zero solutions for the dilaton, axion and the $U(1)$ gauge field.

The expression for energy is given by

$$E = m. \tag{36}$$

This expression also represents the ADM mass of the black hole.

Case II. $|b| \neq |a|$

As is demonstrated in [118], in this situation it is not always possible to construct an analytic closed form black hole solution from the given metric ansatz, as only some special values

enable this scheme. For the string theory the case $a = 1$ and $b \ll 1$ is of importance, and the axion field ξ is trivial up to $O(b)$ ((5.32) in [118]). The metric has a new form

$$ds^2 = \frac{(r - r_+)(r - r_-)}{r^2 - r_0^2} dt^2 - \frac{r^2 - r_0^2}{(r - r_+)(r - r_-)} dr^2 - (r^2 - r_0^2)(d\theta^2 + \sin^2\theta d\varphi^2) \tag{37}$$

and is described by the dilaton charge $Q_\phi = \frac{(Q_m^2 - Q_e^2)e^{-\phi_0}}{m}$ and by the quantities

$$\phi(r) = \phi_0 + \ln\left(\frac{r - r_0}{r + r_0}\right), \quad \xi(r) = \xi_0, \tag{38}$$

$$F_{tr} = \frac{Q_e}{(r + r_0)^2} dt \wedge dr, \quad F_{\theta\varphi} = Q_m \sin\theta d\theta \wedge d\varphi, \tag{39}$$

with

$$r_0 = \frac{(Q_e^2 - Q_m^2)e^{-\phi_0}}{2m}, \quad r_\pm = m \pm \sqrt{m^2 + r_0^2 - (Q_e^2 + Q_m^2)e^{-\phi_0}}. \tag{40}$$

This black hole solution presents two horizons and two charges, electric and magnetic.

The calculations performed with (15), (19) applied to (37) yield the energy in the Møller prescription

$$E(r) = \frac{r-r^2 + r_+r^2 + r_-r_0^2 + r_+r_0^2 - 2r r_0^2 - 2r r_+r_-}{2(r^2 - r_0^2)}. \tag{41}$$

From (38), (39), (40) and (41) it results that the energy distribution depends on the mass m , the total (bare) electric and magnetic charges Q_e and Q_m , r and ϕ_0 .

Using (40) in (41) we obtain

$$E(r) = \frac{8m^3 r^2 + 2m(Q_e^2 - Q_m^2)^2 e^{-2\phi_0} - 8m^2 r(Q_e^2 + Q_m^2)e^{-\phi_0}}{2[4m^2 r^2 - (Q_e^2 - Q_m^2)^2 e^{-2\phi_0}]}. \tag{42}$$

In the special case of $Q_e = 0$ or $Q_m = 0$ combined with the coordinate transformation $r + r_0 \rightarrow r$ the (GHS) [119, 120] magnetically or electrically charged black holes are recovered [118]. For $Q_e = Q_m$ or $Q_e = -Q_m$ one gets a vanishing value for the dilaton charge and this leads to the case of the standard Reissner-Nordstöm black hole solution [118].

(2) We consider now the asymptotically non-flat black hole solutions and we calculate the energy in the Møller prescription for the same cases $|b| = |a|$ and $|b| \neq |a|$ considering some special values.

Case I. $|b| = |a|$

The solutions are given by the value $n = 1/(1 + a^2)$ and $K_2 = 0$, with $r_+ = 2m/(1 + n)$ and $r_- = 0$. The metric is given by

$$ds^2 = \left(\frac{r}{2r_0}\right)^{2n} \left[1 - \frac{2m}{(1-n)r}\right] dt^2 - \left(\frac{2r_0}{r}\right)^{2n} \left[1 - \frac{2m}{(1-n)r}\right]^{-1} dr^2 - r^2 \left(\frac{2r_0}{r}\right)^{2n} (d\theta^2 + \sin^2\theta d\varphi^2). \tag{43}$$

The dilaton field, the axion field and the electromagnetic field strengths F_{tr} and $F_{\theta\varphi}$ are given by (5.45) and (5.46) in [118], with $q^2 = q_e^2 + q_m^2$. This describes a black hole with a causal structure similar to the standard Schwarzschild black hole and with a null hypersurface obtained for $r = 2m/(1 - n)$, which is also the value for which the event horizon is regular.

a. In the case $a = b = 1$ in the Einstein frame the metric has the form

$$ds^2 = \left(\frac{r - 4m}{2r_0}\right) dt^2 - \left(\frac{2r_0}{r - 4m}\right) dr^2 - 2r_0 r (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{44}$$

with the dilaton and axion fields given in (5.48) in [118] and with the electromagnetic field strengths $F_{tr} = q_e/(2q^2) dt \wedge dr$ and $F_{\theta\varphi} = q_m \sin \theta d\theta \wedge d\varphi$. In the special cases $q_e = 0$ or $q_m = 0$ the axion field ξ vanishes and we recover the cases of magnetically or electrically charged dilaton black holes with curved asymptotes [123]. We also notice that asymptotically non-flat magnetically charged dilaton black hole solutions for particular values of the mass m and magnetic charge q_m have been developed [124].

The expression for energy calculated with the Møller definition is given by

$$E(r) = \frac{r}{2}. \tag{45}$$

b. For $a = b \ll 1$ we have $n = 1/(1 + a^2) \approx 1$ and the metric becomes

$$ds^2 = \left(\frac{r}{2r_0}\right)^2 \left[1 - \frac{2m}{a^2 r}\right] dt^2 - \left(\frac{2r_0}{r}\right)^2 \left[1 - \frac{2m}{a^2 r}\right]^{-1} dr^2 - 4r_0^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{46}$$

where the dilaton and axion fields are expressed with (5.54) in [118].

For the energy distribution we obtain

$$E(r) = \frac{1}{2} \left\{ \frac{r}{2r_0^2} \left[1 - \frac{2m}{a^2 r}\right] + \frac{m}{2a^2 r_0^2} \right\} 4r_0^2 = r - \frac{m}{a^2}. \tag{47}$$

The energy distribution presents a dependence on r_0, r , the mass m of the black hole and the parameter a . This expression can be also written

$$E(r) = \frac{1}{2} \left\{ 2r \left[1 - \frac{2m}{a^2 r}\right] + 2m \right\}. \tag{48}$$

In the limit $a \rightarrow 0$ the expression for energy given by (47) diverges. At large distances $r \rightarrow \infty$ the energy distribution tends toward infinity.

c. In the case $a = b \gg 1$ with $n = 1/(1 + a^2) \approx 1/a^2$ the solution is described by the metric

$$ds^2 = \left(\frac{r}{2r_0}\right)^{\frac{2}{a^2}} \left[1 - \frac{2a^2 m}{(a^2 - 1)r}\right] dt^2 - \left(\frac{2r_0}{r}\right)^{\frac{2}{a^2}} \left[1 - \frac{2a^2 m}{(a^2 - 1)r}\right]^{-1} dr^2 - r^2 \left(\frac{2r_0}{r}\right)^{\frac{2}{a^2}} (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{49}$$

with the dilaton and axion fields given in (5.56) in [118] and with the electromagnetic field strengths $F_{1r} \approx q_e/(a^2 q^2)dt \wedge dr$ and $F_{\theta\varphi} = q_m \sin\theta d\theta \wedge d\varphi$.

The corresponding calculations using (15), (19) and (49) lead to the expression for energy which is given by

$$E(r) = \frac{r a^2 - r - 2 a^2 m + a^4 m}{(a^2 - 1)a^2}. \tag{50}$$

In the limit $a \rightarrow \infty$ we recover the energy for the Schwarzschild black hole solution

$$E = m. \tag{51}$$

As in the case $a = b \gg 1$ for the asymptotically flat black hole solutions this expression also represents the ADM mass of the black hole, even if the solution is non-flat asymptotically for a finite value of the parameter a .

Case II. $|b| \neq |a|$

Like in the case of the asymptotically flat black hole solutions is not allowed to develop analytic closed form black hole solutions. We have to take into account special values for the parameters a and b , $a = 1$ and $b \ll 1$ and consider the axion ξ trivial up to order b and written as $\xi = \xi_0 + O(b)$. The solution is developed neglecting the $O(b)$ terms and for $n = 1/2$ and $K_2 = 2 q_e^2 q_m^2 / r_0^2$. The metric corresponds to a dyonic black hole given by

$$ds^2 = \frac{(r - r_+)(r - r_-)}{2 r_0 r} dt^2 - \frac{2 r_0 r}{(r - r_+)(r - r_-)} dr^2 - 2 r_0 r (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{52}$$

described by the quantities

$$\phi(r) = -\ln\left(\frac{q_e^2}{r_0 r}\right), \quad \xi(r) = \xi_0, \tag{53}$$

$$F_{1r} = \frac{1}{2 q_e} dt \wedge dr, \quad F_{\theta\varphi} = q_m \sin\theta d\theta \wedge d\varphi, \tag{54}$$

where

$$r_{\pm} = 2 \left(m \pm \sqrt{m^2 - \frac{q_e^2 q_m^2}{4 r_0^2}} \right). \tag{55}$$

The energy in the Møller prescription is given by

$$E(r) = \frac{1}{2} \frac{r^2 - r_+ r_-}{r} = \frac{1}{2} \frac{r^2 - \frac{q_e^2 q_m^2}{r_0^2}}{r}. \tag{56}$$

In this case the expression for energy depends explicitly on the electromagnetic charges q_e and q_m . It is interesting to notice that in this case the interchange of q_e and q_m does not modify the expression for energy. In the limit cases $r \rightarrow 0$ and $r \rightarrow \infty$ the energy distribution diverges.

4 Discussion

We calculate the energy and momentum distributions in the Møller prescription for some asymptotically flat and asymptotically non-flat solutions in the context of Einstein-Maxwell-dilaton-axion theory [118]. It is important to emphasize that using the Møller energy-momentum complex the requirement of performing the calculations in quasi-Cartesian coordinates can be avoided.

We consider two special cases $|b| = |a|$ and $|b| \neq |a|$. For $|b| = |a|$ we investigate the corresponding three special values for the parameters a and b , which are $a = b = 1$, $a = b \ll 1$ and $a = b \gg 1$ and some limit cases. In the case $|b| \neq |a|$ the special values $a = b \ll 1$ yield important results and also some particular cases are studied. For all the particular cases mentioned the momenta are found to be zero.

In the Tables 1 and 2 we briefly present the expressions for energy obtained in the case of asymptotically flat and asymptotically non-flat solutions, respectively, and some limit cases that occur in each situation.

Firstly, we outline the results for the case of asymptotically flat black hole solutions.

Now, we point out the results obtained for the asymptotically non-flat black hole solutions.

The expression for energy $E(r) = m - \frac{Q^2}{r}$ obtained in the case of asymptotically flat black hole solutions for $|b| = |a|$, $a \ll 1$, limit case $a \rightarrow 0$ and $|b| \neq |a|$, $a = 1$ and $b \ll 1$, $Q_e = Q_m$ or $Q_e = -Q_m$, respectively is in good agreement with the result given by Komar [125].

All these results illustrate that the use of the Møller prescription for the evaluation of the expressions for energy is an important option. We notice that interesting particular cases arise for both classes of solutions AF and ANF, respectively.

For future work, we intend to explore the results yielded by the pseudotensorial method for these black hole solutions using other energy-momentum complexes.

Table 1 The case of asymptotically flat black hole solutions

Case	Energy distribution
$ b = a $	$E(r) = \frac{m_0 a^2 r + m_0 r - 4 m_0 r_0 - r_0 a^2 r + r_0 r}{r(a^2 + 1)}$
$ b = a , a = b = 1$	$E(r) = m(1 - \frac{2r_0}{r}) = m - \frac{Q^2 e^{-\phi_0}}{r}$
$ b = a , a = b = 1, Q_e = 0, Q_m = Q$	$E(r) = m - \frac{Q_m^2 e^{-\phi_0}}{r}$ (GHS)
$ b = a , a = b = 1, Q_m = 0, Q_e = Q$	$E(r) = m - \frac{Q_e^2 e^{-\phi_0}}{r}$ (GHS)
$ b = a , a = b \ll 1$, limit case $a \rightarrow 0$	$E(r) = m - \frac{Q^2}{r}$ standard dyonic RN black hole
$ b = a , a = b \gg 1$, limit case $a \rightarrow \infty$	$E = m$ standard Schwarzschild black hole
$ b \neq a , a = 1$ and $b \ll 1$	$E(r) = \frac{8m^3 r^2 + 2m(Q_e^2 - Q_m^2)^2 e^{-2\phi_0} - 8m^2 r(Q_e^2 + Q_m^2) e^{-\phi_0}}{2[4m^2 r^2 - (Q_e^2 - Q_m^2)^2 e^{-2\phi_0}]}$
$ b \neq a , a = 1$ and $b \ll 1, Q_e = 0$	$E(r) = \frac{8m^3 r^2 + 2mQ_m^4 e^{-2\phi_0} - 8m^2 rQ_m^2 e^{-\phi_0}}{2[4m^2 r^2 - Q_m^4 e^{-2\phi_0}]}$ (GHS)
$ b \neq a , a = 1$ and $b \ll 1, Q_m = 0$	$E(r) = \frac{8m^3 r^2 + 2mQ_e^4 e^{-2\phi_0} - 8m^2 rQ_e^2 e^{-\phi_0}}{2[4m^2 r^2 - Q_e^4 e^{-2\phi_0}]}$ (GHS)
$ b \neq a , a = 1$ and $b \ll 1, Q_e = Q_m$ or $Q_e = -Q_m$	$E(r) = m - \frac{2Q^2 e^{-\phi_0}}{r}$ standard RN black hole

Table 2 The case of asymptotically non-flat black hole solutions

Case	Energy distribution
$ b = a , a = b = 1$	$E(r) = \frac{r}{2}$
$ b = a , a = b \ll 1$	$E(r) = r - \frac{m}{a^2}$
$ b = a , a = b \ll 1, \text{ limit case } a \rightarrow 0$	$E(r) \rightarrow -\infty$
$ b = a , a = b \ll 1, \text{ limit case } r \rightarrow \infty$	$E(r) \rightarrow \infty$
$ b = a , a = b \gg 1$	$E(r) = \frac{r a^2 - r - 2a^2 m + a^4 m}{(a^2 - 1)a^2}$
$ b = a , a = b \gg 1, \text{ limit case } a \rightarrow \infty$	$E = m$ standard Schwarzschild black hole
$ b \neq a , a = 1 \text{ and } b \ll 1$	$E(r) = \frac{1}{2} \frac{r^2 - r + r -}{r} = \frac{1}{2} \frac{r^2 - \frac{q_e^2 q_m^2}{r_0^2}}{r}$
$ b \neq a , a = 1 \text{ and } b \ll 1, \text{ limit case } r \rightarrow 0$	$E(r) \rightarrow \pm\infty$
$ b \neq a , a = 1 \text{ and } b \ll 1, \text{ limit case } r \rightarrow \infty$	$E(r) \rightarrow \infty$

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